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# Griffiths' Singularities in Diluted Ising Models on the Cayley Tree

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The Griffiths singularities are fully exhibited for a class of diluted ferromagnetic Ising models defined on the Cayley tree (Bethe lattice). For the deterministic model the Lee–Yang circle theorem is explicitly proven for the magnetization at the origin and it is shown that, in the thermodynamic limit, the Lee–Yang singularities become dense in the entire unit circle for the whole ferromagnetic phase. Smoothness (infinite differentiability) of the quenched magnetization $m$ at the origin with respect to the external magnetic field is also proven for convenient choices of temperature and disorder. From our analysis we also conclude that the existence of metastable states is impossible for the random models under consideration.

**KEY WORDS:** Lee–Yang singularities; Griffiths' singularities; infinite differentiability; metastable states.

## 1. INTRODUCTION

For systems with quenched impurities the equilibrium properties depend not only on the usual parameters (temperature, magnetic field, etc.) but also on the impurity distribution. The presence of randomness may affect their critical behavior by rounding first order transitions [AW] and may also produce singularities of unconventional type: a whole region of the phase diagram corresponding to a pure phase may be populated by "weak" singularities (see e.g. ref. [F]).

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Griffiths [G] was the first to realize and actually prove such phenomenon. He considered a random Ising ferromagnet in $\mathbb{Z}^d$ described by the Hamiltonian

$$\mathscr{H}(\sigma; \xi) := - \sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y - H \sum_x \sigma_x \tag{1.1}$$

where $\sigma: \mathbb{Z}^d \ni x \mapsto \sigma_x \in \{1, -1\}$ is a configuration of spins which are coupled by a nearest neighbor interaction of the form

$$J_{xy} = \xi_x \xi_y \tag{1.2}$$

with $\xi = \{\xi_x, x \in \mathbb{Z}^d\}$ being independent and equally distributed Bernoulli random variables

$$\xi_x = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1 - p \end{cases} \tag{1.3}$$

Griffiths has shown [G] that the free energy for this model cannot be analytically continued from $H > 0$ to $H < 0$ for all temperatures below the critical temperature $T_c(p = 1)$ of the deterministic model and all $p < p_c$, the threshold of site percolation. This result has been extended beyond $p_c$, independently, by Sütő [S] and Fröhlich [F]. As a consequence, Sütő claimed that a metastable phase cannot take place in diluted Ising models.

Although the presence of Griffiths' singularities is now recognized to be a common feature of disordered systems (see ref. [F] and references therein), few progress has been made since Griffiths' original work. The emergence of Griffiths' singularities may be explained as follows. The fact that there occurs, with positive probability, arbitrarily large regions inside which the system is strongly correlated, leads the Lee–Yang's zeros [LY] to become dense over the unit circle in the complex plane. It is this very last statement that prevents the free energy to be continued analytically and which turns out to be hard to verify even in the most simple examples.

The aim of this article is to exhibit in all possible details the Griffiths' singularities in a caricature model for the system (1.1)–(1.3). We consider a class of bond diluted Ising models given by the Hamiltonian (1.1) defined on the Bethe lattice whose disorder is introduced as follows.

Let $\mathscr{C}_k$ be an homogeneous rooted Cayley tree (Bethe lattice) of order $k$, i.e., a tree with $k + 1$ bonds attached to each site which is not the root (called origin).

A bond $b$ is classified according to its distance from the origin. Thus, $b$ belongs to the generation $M$ (or to the equivalent class indexed by $M$)

if a directed path starting from the origin pass exactly by $M-1$ bonds before it reaches $b$.

To each generation $M$ a Bernoulli random variable $\xi_M$

$$\xi_M = \begin{cases} 1, & \text{with probability } p_M \\ 0, & \text{with probability } q_M = 1 - p_M \end{cases} \tag{1.4}$$

is assigned and we then set $J_{xy} = \xi_M$ if $\langle xy \rangle \equiv b$ is a bond at the generation $M$ and 0 otherwise. See Fig. 1.

For any event $A$, which is an element of the $\sigma$-algebra generated by the space of spin configurations, we define the quenched expected value of $A$ as

$$\bar{A} \equiv \mathbb{E}[A] = \mathbb{E}_\xi \langle A \rangle(\xi) \tag{1.5}$$

where $\mathbb{E}_\xi$ is the expectation with respect to the disorder variables and

$$\langle A \rangle(\xi) = \frac{1}{Z(\xi)} \sum_\sigma A(\sigma)\, e^{-\beta \mathscr{H}(\sigma;\, \xi)} \tag{1.6}$$

is the thermal average with

$$Z(\xi) = \sum_\sigma e^{-\beta \mathscr{H}(\sigma;\, \xi)}$$

being the partition function.

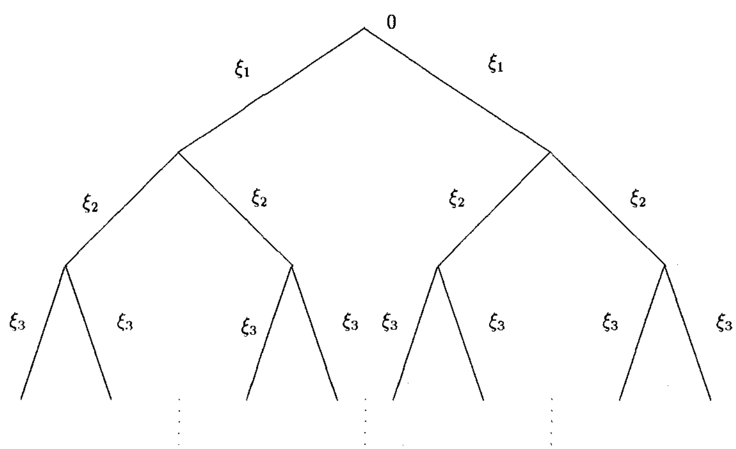


Fig. 1. The homogeneous rooted Cayley tree for $k = 2$ and the couplings $\xi_M$.

The expectation (1.5) is a function of the inverse temperature $\beta$, the external magnetic field $H$ and the sequence of Bernoulli parameters $\mathbf{p} = \{p_j\}_{j=1}^{\infty}$. Let $\pi = \{\mathbf{p}: 0 < p_j \leqslant 1, j = 1, 2,...\}$ and notice that $\prod_j p_j \leqslant 1$. For $0 \leqslant a \leqslant 1$, let $\pi_a = \{\mathbf{p} \in \pi: \lim_{n \to \infty} p_1 \cdots p_n = a\}$. Also, for convenience, we write $\zeta = e^{-2\beta}$ and $z = e^{-2\beta H}$.

We shall first take the expectation in a finite tree $\mathscr{C}_{k, M}$, $M = 1, 2,...$, with free boundary conditions and study the function $\bar{A}$: $[0, 1] \times \mathbb{C} \times \pi \mapsto \bar{A}(\zeta, z, \mathbf{p}) \in \mathbb{C}$ in the limit as $M$ goes to infinite. We recall that $\bar{A}$ is not translation invariant, even at the thermodynamic limit.

As in ref. [G], instead of analyzing the free energy, we shall be concerned with the magnetization at origin $m := \mathbb{E}[\sigma_0] = \mathbb{E}_\xi \langle \sigma_0 \rangle (\xi)$ which can be written as

$$m = F + I \tag{1.7}$$

with $F := \lim_{N \to \infty} F_N$, where

$$F_N := \sum_{M=0}^{N-1} a_M \langle \sigma_0 \rangle_M (1) \tag{1.8}$$

gives the contribution to $m$ due to finite clusters and $I := \lim_{N \to \infty} I_N$, where

$$I_N := p_1 \cdots p_n \langle \sigma_0 \rangle_N (1) \tag{1.9}$$

is the contribution of the (unique) infinite cluster. Here, for any observable $A$, $\langle A \rangle_M$ is the thermal average (1.6) with $\mathscr{C}_k$ replaced by the finite tree $\mathscr{C}_{k, M}$ and $a_M$, $M \in \mathbb{N}$, is given by

$$a_0 := 1 - p_1,$$
$$a_N := p_1 \cdots p_N (1 - p_{N+1}), \qquad N \geqslant 1$$

Equation (1.7) is a consequence of the fact that $\langle \sigma_0 \rangle (\xi) = \langle \sigma_0 \rangle_M (1)$ for all $\xi$ such that $\xi_1 = \cdots = \xi_M = 1$ and $\xi_{M+1} = 0$. In addition, since

$$\sum_{n=0}^{N-1} a_n + p_1 \cdots p_N = 1$$

for any $N \in \mathbb{N}$, we note that $\sum_{n=0}^{\infty} a_n = 1 - a$ for all $p \in \pi_a$. Clearly, $\{a_n\}_{n=0}^{\infty}$ is a summable sequence.

There are two different manners of $m$ fail to be analytic in $z$ depending on the behavior of $F$ and $I$. When $\zeta > \zeta_c$, with $\zeta_c$ being the critical point of the deterministic model, both $F$ and $I$ are analytic in a neighborhood of $\mathbb{R}_+$. $F$ is responsible for the Griffiths' singularities for all $\zeta < \zeta_c$ and $\mathbf{p} \in \pi$ while $I$ develops a discontinuity at $z = 1$ for $\zeta < \zeta_c$ and $\mathbf{p} \in \pi_a$ with $a > 0$ (see Theorem 1.6 for precise statements).

By assigning a single random variable to bonds at the same generation, only thermal expectations of "perfect" finite trees have to be computed in the quenched magnetization (1.7). This simplified form of $m$, which is analogous to that introduced by McCoy and Wu [MW] for the random Ising model in $\mathbb{Z}^2$ (see also ref. [Fi] for a one dimensional model related to ours), has to be compared with similar equation for the model defined by Eqs. (1.1)–(1.3):

$$m = \sum_{\psi} P_{\psi} m_{\psi} \tag{1.10}$$

Here, $m_{\psi}$ is the magnetization density in the cluster $\psi$ (a finite connected piece of $\mathbb{Z}^d$) and $P_{\psi}$ the probability that an isolated cluster $\psi$ contains the origin. The sum in (1.10) contains the contributions of finite and infinite clusters. The sum over finite clusters runs over $c^n$ different clusters $\psi$ connected to the origin and with size $|\psi| = n$, $n \in \mathbb{N}$, whereas only one cluster with size $(k^{M+1} - 1)/(k-1)$ for each generation $M$ of the Cayley tree $\mathscr{C}_k$ contributes to (1.7). Also, only one infinite cluster appears in the class of models we considered.

On a general ground the Griffiths singularities emerge from the contribution of the finite clusters to $m$, provided there exists an infinite collection $C$ of finite clusters such that the union $U(C)$ of all Lee–Yang zeros of all clusters in $C$ has $z = 1$ as an accumulation point. For the absence of metastability $\overline{U(C)} = S^1$ should be required. To fully exhibit the presence of Griffiths singularities or the absence of metastability in random ferromagnetic models the validity of such requirements on $U(C)$ has to be proven explicitly. For this, the relation between $n \in \mathbb{N}$ and the number of clusters of size $n$ connected to the origin is of lesser importance and, hence, the class of models we considered is likely to share some important properties of more complicated systems.

In this paper we will prove the following two theorems, which illustrate explicitly Lee–Yang and Griffiths theorems for the particular model introduced above. In order to simplify the presentation of the results of this work, our analysis will be restricted to the Cayley tree $\mathscr{C}_k$ with $k = 2$.

**Theorem 1.1** (Analyticity). The magnetization at origin, $m = m(\zeta, z, \mathbf{p})$, given by (1.7) is an analytic function of $z$ in the domain $\{z \in \mathbb{C}, |z| < 1\} \cup \{z \in \mathbb{C}, |z| > 1\}$ for all $\zeta \in [0, 1]$ and $\mathbf{p} \in \pi$. ∎

Griffiths [G] has argued that the Lee–Yang singularities, in the thermodynamic limit, accumulate at $z=1$ for all $T<T_c$ (see also ref. [KG]). The next theorem shows that the singularities in our model are dense over the arc $A_K := \{z=e^{i\theta}\colon K \leqslant |\theta| \leqslant \pi\}$ in the unit circle for some $K=K(\zeta)$ and do accumulate at $z=1$ for all $\zeta \leqslant \zeta_c \equiv 1/3$. More precisely,

**Theorem 1.2** (Distribution of Singularities). Let $\zeta \in [0, 1]$ and $\mathbf{p} \in \pi$ such that $a_n \neq 0$ for infinitely many values of $n$. Then, the magnetization at origin, $m = m(\zeta, z, \mathbf{p})$, cannot be analytically continued from $\{|z|<1\}$ to $\{|z|>1\}$ (or vice-versa) along the arc of circle $A_K$ where $K=K(\zeta)$ is given by

$$K(\zeta) := \begin{cases} 3 \arccos\left(\dfrac{3-8\zeta+3\zeta^2}{2\zeta}\right) & \\ \qquad - 4 \arctan\left(\dfrac{\sqrt{3(3-\zeta)(3\zeta-1)}}{5-3\zeta}\right) & \text{if } \zeta_c < \zeta < \zeta_l \\ 0 & \text{if } \zeta \leqslant \zeta_c \end{cases}$$

with $\zeta_l \in (1/3, 1)$, $\zeta_l \simeq 0.46409$.

In addition, for all $\zeta > \zeta_c$ and $\mathbf{p} \in \pi$, $m$ is an analytic function of $z$ for all $z \in \mathbb{C} \backslash A_\kappa$, where $\kappa = \kappa(\zeta)$ is given by

$$\kappa := 4 \arctan\left(\frac{\zeta \sin \bar{\phi}}{1+\zeta \cos \bar{\phi}}\right) - \bar{\phi} \tag{1.11}$$

with $\cos \bar{\phi} := 1/(2\zeta) - \frac{3}{2}\zeta$. ∎

**Remark 1.3.** The proof of Theorem 1.2 is based on a detailed description of the singularities of $\langle \sigma_0 \rangle_M(1)$ which are located, by Theorem 1.1, in the unit circle. These are isolated poles which accumulate in the arc $A_K$. The measure properties of this set will not be described in this work (for this and some of our results in a similar model see ref. [BG-R]–[Mo]). Our estimates can be easily adapted to provide better values for $K$ and $\zeta_l$. We believe that $\zeta_l$ can be pushed up to 1. Note that $A_\kappa \backslash A_K \neq \varnothing$ for all $\zeta_c < \zeta < \zeta_l$ with $\lim_{\zeta \searrow \zeta_c} K(\zeta) = \lim_{\zeta \searrow \zeta_c} \kappa(\zeta) = 0$ (see Fig. 2).

**Remark 1.4.** Due to the dilution, free boundary conditions were imposed. It is interesting to note that the circle theorem is no longer valid for boundary conditions corresponding to a different magnetic field (eventually infinite) acting on the spins at the boundary.

**Remark 1.5.** As discussed by Sütő, Theorem 1.2 immediately implies absence of metastability for our diluted model. We recall that our

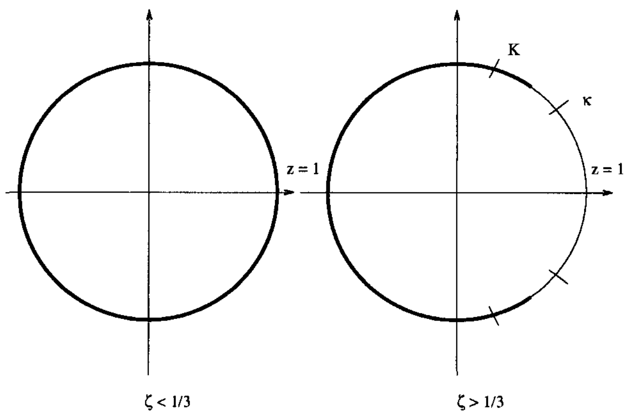


Fig. 2. The limit set of singularities (bold arcs) in two regimes.

models distinguish the contribution of the finite clusters from the contribution of the infinite cluster. When $\mathbf{p} = 1$ the system reduces to the deterministic Ising ferromagnet on the Cayley tree which exhibits metastability as in the mean field theory.

Recently, Dreifus, Klein and Perez [DKP] have established infinite differentiability of the quenched magnetization for general disordered spin systems using a modified high temperature expansion which converges in the presence of Griffiths' singularities. We here are able to study the right and left derivatives of $m$ at $z = 1$ when spontaneous magnetization occurs and the high temperature expansion does not converge. In view of Theorems 1.1 and 1.2 it suffices to examine only the neighborhood of $z = 1$ in $z \in \mathbb{R}_+$ when $\zeta < \zeta_c$.

**Theorem 1.6** (Continuity and Differentiability). For any $0 \leqslant \zeta < \zeta_c$, $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi_0$, the quenched magnetization $m$ given by (1.7) is always a continuous function of $z$ with $m(1) = 0$. Moreover, $F$ is at least $k$-times differentiable at $z = 1$ for $0 < \zeta < \zeta_c$ and $\mathbf{p} \in \pi$, provided the sequence $\{a_j \lambda^{kj}\}_{j=0}^{\infty}$ with

$$\lambda = \lambda(\zeta) := 2\,\frac{1-\zeta}{1+\zeta}$$

is summable, i.e.,

$$\frac{1}{k!}\left|\frac{d^k F}{dz^k}\right| \leqslant C D^k \sum_{j=0}^{\infty} \lambda^{kj} a_j \tag{1.12}$$

for some finite constants $C = C(\zeta)$ and $D = D(\zeta)$.

In addition, for all $\mathbf{p} \in \pi_a$, with $a > 0, m$ is a discontinuous function of $z$ at $z = 1$ with

$$\lim_{\varepsilon \to 0} [m(1-\varepsilon) - m(1+\varepsilon)] = 2a \frac{(1-3\zeta)^{1/2}(1+\zeta)^{1/2}}{1-\zeta} \quad \blacksquare \tag{1.13}$$

**Remark 1.7.** For $z \in \mathbb{R}_+$, $z \neq 1$, (1.12) holds with $\lambda^k$ replaced by $\mu = \mu(\zeta, z)$, uniformly in $k$ (see Theorem 6.7). This, in particular, implies that $m(z)$ has a convergent power series, in agreement with Theorem 1.1.

**Remark 1.8.** The summability condition on $\{a_j \lambda^{jk}\}_{j=0}^{\infty}$ can be fulfilled in two different regimes:

1. When $\lim_{z \to 1} m(z) = 0$, the Bernoulli parameters $\mathbf{p} \in \pi_0$, can be chosen such that

$$\frac{-1}{N} \sum_{j=1}^{N} \ln p_j > k \ln \lambda + \rho$$

holds for all $N > N_0$, for some $N_0 = N_0(k, \lambda) < \infty$ and $\rho > 0$. Notice that, if $p_j \sim e^{-j^\varepsilon}$ with $\varepsilon > 0$, the magnetization $m$ is infinitely differentiable.

2. For $\mathbf{p} \in \pi_a$, with $a > 0$, the function $I$ must be considered. One can show, by applying the implicit function theorem to Eq. (3.18), that $I(z)$ can be analytically continued from $z < 1$ to $z > 1$ (and vice-versa). From this and Theorem 1.6, if $p_{j+1} > 1 - (\delta \lambda^{-k})^j$, with $\delta < 1$, then $\lambda^{jk} a_j < \delta^j$ and the right and left derivatives of $m$ at $z = 1$ exist up to order $k$. If $p_j \sim 1 - e^{-j^{1+\varepsilon}}$, with $\varepsilon > 0$, the right and left derivatives of all orders exist.

Lee–Yang circle theorem (Theorem 1.1) is proven in Section 3, Griffiths theorem (Theorem 1.2) in Sections 4 and 5 and Theorem 1.6 will be proven in Section 6.

## 2. SOME BASIC RESULTS AND DEFINITIONS

In this section we present some basic mathematical facts which we will use in the proofs of our main results. Some useful definitions are also introduced.

For $z \in \mathbb{C}$ consider the map $\tau_z \colon \mathbb{C} \mapsto \mathbb{C}$ given by $\mathbb{C} \ni u \to \tau_z(u)$ with

$$\tau_z(u) := h(zu)$$

where, for $w \in \mathbb{C}$,

$$h(w) := \left( \frac{\zeta + w}{1 + \zeta w} \right)^2 \tag{2.1}$$

with $0 \leqslant \zeta \leqslant 1$.

Let us denote by $\tau_z^{(n)}$ the $n$th composition of $\tau_z$ with itself, for $n \in \mathbb{N}$:

$$\tau_z^{(n)} := \underbrace{\tau_z \circ \cdots \circ \tau_z}_{n} \tag{2.2}$$

with $\tau_z^{(0)}(u) = u$, $u \in \mathbb{C}$. Very often we will be interested in the following functions of $z$:

$$w_n(z) := z\tau_z^{(n)}(1) \tag{2.3}$$

with $n \in \mathbb{N}$.

Let us start with some elementary remarks concerning the function $h$. First, note that $h$ has a unique double pole at $w = -1/\zeta$. Moreover, $h(w)$ is bounded on $\mathbb{C} \setminus \mathcal{O}$, where $\mathcal{O}$ is any open set containing the pole and has a removable singularity at the complex infinity. Therefore it could actually be considered as a map of the Riemann sphere $S^2$ into itself. As such a map $h$ is holomorphic in $S^2 \setminus \{-1/\zeta\}$. Moreover, in $S^2$ one has $h(\bar{w}) = \overline{h(w)}$ and

$$h(w^{-1}) = h(w)^{-1} \tag{2.4}$$

Both relations together imply that $h$ maps the unit circle $S^1 := \{w \in S^2 : |w| = 1\}$ into itself, since for $w \in S^1$, $\overline{h(w)} = h(w)^{-1}$.

Let us denote by $D_{<1}$ the open unit disk in $S^2$ and by $D_{>1}$ the complement of its closure. More generally, for $a > 0$ call

$$D_{<a} := \{w \in S^2 : |w| < a\} \qquad \text{and} \qquad D_{>a} := \{w \in S^2 : |w| > a\}$$

The following theorem is very important for this work.

**Theorem 2.1.** For the whole interval $0 \leqslant \zeta < 1$, the function $h$ maps $D_{<1}$ into itself and $D_{>1}$ into itself. ▌

*Proof.* By (2.4) it is enough to prove the claim for $D_{<1}$. For $0 \leqslant \zeta < 1$ the function $h$ is analytic in $D_{<1}$. If $w \in S^1$, the boundary of $D_{<1}$, we have seen that $|h(w)| = 1$. Therefore, since $h$ is not constant in $D_{<1}$, we conclude by the Maximum Modulus Theorem (see e.g., ref. [T]) that $|h(w)| < 1$ for all $w \in D_{<1}$. ▌

One can also trivially see that $h$ maps $\mathbb{R}_+$ into itself. So, we have also established the following picture, which will play a central role in our discussions:

**Corollary 2.2.** As a function of $z$, $w_n(z)$ maps each of the sets $D_{<1}$, $S^1$ and $D_{>1}$ into itself for all $n \in \mathbb{N}$ and all $\zeta \in [0, 1)$. Moreover, each of the sets $[0, 1)$ and $(1, \infty) \subset \mathbb{R}_+$ is also mapped into itself. Finally, $w_n(1) = 1$, for all $n \in \mathbb{N}$. ▌

We will also make use of the following observation. The function $h$ is the ratio of two polynomials in $z$ of degree 2 and, hence, $\tau_z^{(n)}(1)$ can be written as the ratio of two polynomials of degree $2^{n+1} - 2$:

$$\tau_z^{(n)}(1) = \frac{P_n(z)}{Q_n(s)} \tag{2.5}$$

where $P_n(z)$ and $Q_n(z)$ are polynomials in $z$ of degree $2^{n+1} - 2$.

## 3. ANALYTICITY OF $m$

This section is devoted to the proof of the analyticity of the quenched magnetization at the origin, as described in Theorem 1.1. We start by computing the partition function $Z_M(\zeta)$ in a finite tree with $M$ generations and $\xi \equiv 1$ (deterministic case).

Let $Z_j = (Z_j^+, Z_j^-)$, $j = 0, 1, ..., M$, be a sequence of two-component vectors defined recursively by

$$\begin{aligned} Z_{j-1}^{\sigma} &:= \left( \sum_{\sigma' = \pm 1} e^{\beta\sigma\sigma'} e^{\beta h \sigma'} Z_j^{\sigma'} \right)^2 \\ &= (\zeta z)^{-1} \left( \zeta^{(1-\sigma)/2} Z_j^+ + \zeta^{(1+\sigma)/2} z Z_j^- \right)^2 \end{aligned} \tag{3.1}$$

with $Z_M^+ = Z_M^- = 1$.

If the spin configurations are summed starting from the branches towards the root, the partition function $Z_M(\xi)$ at $\xi \equiv 1$ can be written as

$$Z_M(1) = z^{-1/2} Z_0^+ + z^{1/2} Z_0^- \tag{3.2}$$

To compute the one point-function

$$\langle \sigma_0 \rangle_M(\xi) = \frac{1}{Z_M(\xi)} \sum_{\sigma} \sigma_0 e^{-\beta H(\sigma; \xi)} \tag{3.3}$$

we repeat the calculation leading to (3.2) for the numerator in (3.3). Except by the last summation on the spin at the root, all remaining ones give exactly the same expressions. We thus have

$$\langle\sigma_0\rangle_M(1)=\frac{z^{-1/2}Z_0^+ - z^{1/2}Z_0^-}{z^{-1/2}Z_0^+ + z^{1/2}Z_0^-}\equiv\frac{1-z\Delta_0}{1+z\Delta_0} \tag{3.4}$$

where $\Delta_j = Z_j^- / Z_j^+$.

From (3.1), the sequence $\{\Delta_j\}_{j=0}^{M}$ satisfies the following recurrence relation:

$$\Delta_{j-1}=\tau_z(\Delta_j), \qquad j=1,..., M \tag{3.5}$$

with $\Delta_M = 1$. We have thus proven the following proposition.

**Proposition 3.1.** The one point function $\langle\sigma_0\rangle_M(1)$, defined on a finite tree $\mathscr{C}_{2,M}$ with no disorder ($\xi\equiv 1$), can be written as

$$\langle\sigma_0\rangle_M(1)=\frac{1-w_M(z)}{1+w_M(z)}=r(w_M(z)) \tag{3.6}$$

where

$$r(x):=\frac{1-x}{1+x} \tag{3.7}$$

for $x\in\mathbb{C}$, $x\neq -1$. ∎

We will present the proof of Theorem 1.1 in the next two subsections. The first is dedicated to the proof of analyticity of $F$ and the second to the proof of analyticity of $I$.

## 3.1. Analyticity of $F$ in $S^2\backslash S^1$

Using (3.6) we can write (1.8) as

$$F_N(z):=\sum_{n=0}^{N-1} a_n r(w_n(z)) \tag{3.8}$$

Consider now any open set $\mathscr{B}$ such that $\bar{\mathscr{B}}\subset D_{<1}$. One has

$$\inf_{z\in\mathscr{B}} |w_n(z)+1| > 1-\sup\{|w|,\ w\in\mathscr{B}\} =: C^{-1} > 0 \tag{3.9}$$

since, by Corollary 2.2, $w_n(z) \in D_{<1}$ for all $z \in \mathscr{B}$ and all $n \in \mathbb{N}$. Moreover, under the same hypotheses $|1 - w_n(z)| \leqslant 2$. Therefore, using the fact that $\{a_n\}_{n=0}^{\infty}$ is a summable sequence, we have

$$|F_M(z) - F_N(z)| \leqslant 2C \sum_{n=N}^{M-1} |a_n| < \varepsilon \tag{3.10}$$

for any prescribed $\varepsilon > 0$, by choosing $M > N$, and $N$ large enough. Hence, $F_N(s)$ is a uniform Cauchy sequence of analytic functions in $\mathscr{B}$ and, consequently, its limit exists and is analytic in $\mathscr{B}$.

Since $\mathscr{B}$ is generic, this shows that $F(z)$ is analytic in the whole set $D_{<1}$. To prove that $F(z)$ is also analytic in $D_{>1}$ we observe that, by (2.4), one has

$$w_n(z) = (w_n(z^{-1}))^{-1} \tag{3.11}$$

and, hence, $F_N(z) = -F_N(1/z)$, $N \in \mathbb{N}$.

### 3.2. Analyticity of $I$ in $S^2 \backslash S^1$

Let us consider the closed set $D_1 := D_{<1} \cup S^1 \subset \mathbb{C}$ and the map $\tau_z : D_1 \mapsto D_1$, for $z \in D_{<1}$ fixed.

**Proposition 3.2.** The map $\tau_z$ is contractive on $D_1$ for $0 \leqslant \zeta \leqslant 1$ and $|z| < \rho_-(\zeta)$, where $\rho_-(\zeta)$: $[0, 1] \to [0, 1]$ is given in (3.16). Therefore, by the Banach fixed point theorem, the sequences $\{\tau_z^{(n)}(w)\}_{n \in \mathbb{N}}$, with $w \in D_1$, converge to $\tau \in D_1$ which is the unique solution in $D_1$ of the fixed-point equation

$$\tau = \tau_z(\tau) = h(z\tau) \tag{3.12}$$

Moreover, for $|z| \leqslant \rho_0 < \rho_-(\zeta)$ and $|w|$, $|v| \leqslant 1$ one has

$$|h(zw) - h(zv)| \leqslant q_Q |w - v| \tag{3.13}$$

with $q_Q := 2(1 - \zeta^2)\, \rho_0/(1 - \zeta\rho_0)^2 < 1$. ∎

*Proof.* We want to analyze $|\tau_z(w) - \tau_z(v)| = |h(zw) - h(zv)|$ for $w, v \in D_1$. A computation shows that $h(zw) - h(zv) = Q(z, w, v)(w - v)$, where

$$Q(z, w, z) := \left[\frac{\zeta + zw}{1 + \zeta zw} + \frac{\zeta + zv}{1 + \zeta zv}\right] \frac{z(1 - \zeta^2)}{(1 + \zeta zw)(1 + \zeta zv)} \tag{3.14}$$

We prove that $\tau_z$ is contractive provided we find conditions which guarantee that $|Q(z, w, v)| < q$ for some constant $q$ with $0 \leqslant q < 1$.

Since $|(\zeta + zw)/(1 + \zeta zw)| \leqslant 1$ for $|z|$, $|w| \leqslant 1$ one has

$$|Q(z, w, v)| \leqslant 2(1-\zeta^2) \frac{|z|}{(1-\zeta\,|z|)^2} \tag{3.15}$$

One has $|Q(z, w, v)| < 1$ provided

$$-\zeta^2\,|z|^2 + 2\,|z|\,(1+\zeta-\zeta^2) - 1 = -\zeta^2(|z| - \rho_-(\zeta))(|z| - \rho_+(\zeta)) < 0$$

where

$$\rho_\pm(\zeta) := \frac{1+\zeta-\zeta^2 \pm \sqrt{(1+2\zeta-\zeta^2)(1-\zeta^2)}}{\zeta^2} \tag{3.16}$$

Note that $1 + 2\zeta - \zeta^1 \geqslant 0$ for $\zeta \in [0, 1]$.

Next we show that $\rho_+(\zeta) > 1$ and $\rho_-(\zeta) < 1$, for $\zeta \in (0, 1)$. These conditions are equivalent to the condition

$$(1+\zeta-2\zeta^2)^2 - (1+2\zeta-\zeta^2)(1-\zeta^2) < 0 \tag{3.17}$$

but the left hand side equals $-\zeta^2(1-\zeta)(1+3\zeta)$ which is manifestly negative for $0 < \zeta < 1$.

We have established that, under the hypotheses, $|\tau_z(w) - \tau_z(v)| < q\,|w - v|$ for some fixed $q$ with $0 < q < 1$ what means that $\tau_z$ is contractive.

Since the right hand side of (3.15) is an increasing function of $|z|$ for $0 \leqslant |z| < 1/\zeta$, we have $|Q(z, w, v)| \leqslant q_Q < 1$ for $0 \leqslant |z| < \rho_0 < \rho_-(\zeta)$. This completes the proof of the proposition. ∎

**Remark 3.3.** One has to show that $\rho_-(\zeta) > 0$, what means that the region of validity for $z$ of the last proposition is in fact non-empty. Since $1 + \zeta - \zeta^2 \geqslant 0$ for $\zeta \in [0, 1]$, it is enough to show that $(1+\zeta-\zeta^2)^2 - (1+2\zeta-\zeta^2)(1-\zeta^2) \geqslant 0$. But the left-hand side equals $\zeta^2$ and so, positivity of $\rho_-(\zeta)$ is proven. Actually, one finds numerically that $\rho_-(\zeta) > \rho_0 \simeq 0.41$ for all $\zeta \in [0, 1]$. Finally we notice that $\lim_{\zeta \to 0} \rho_-(\zeta) = 1/2$ and that $\rho_-(1) = 1$. This last fact means that the region $\{z \in \mathbb{C} \colon |z| < \rho_-(\zeta)\}$ converges to $D_{<1}$ when $\zeta \to 1$.

We are now in position to establish the following important result:

**Theorem 3.4.** The sequence $w_n(z)$, $n \in \mathbb{N}$, of analytic functions on $D_{<1}$ converges uniformly to an analytic function $w = w(z)$ on the whole set $D_{<1}$. ∎

*Proof.* The sequence $\tau_z^{(n)}(1)$, $n \in \mathbb{N}$, is a sequence of analytic functions of $z \in D_{<1}$ which is uniformly bounded in this domain, since $|\tau_z^{(n)}(1)| < 1$. Proposition 3.2 says that this sequence converges on the open subset $D_{\rho_-} \subset D_{<1}$. Now, according to Vitali's Convergence Theorem (see, e.g., ref. [T]), these facts guarantee that the sequence converges uniformly on the whole $D_{<1}$ to a function $\tau(z)$, analytic on $D_{<1}$. Since $w_n(z) = z\tau_z^{(n)}(1)$, the sequence of analytic functions $w_n(z)$ converges uniformly to $w(z) = z\tau(z)$. ▌

**Theorem 3.5.** For all $\zeta \in (0, 1]$ the function $\tau = \tau(z)$ fulfils the fixed point equation $\tau = h(z\tau)$ on the whole set $D_{<1}$. As a consequence, $\tau(z)$ has no zeros on $D_{<1}$. ▌

*Proof.* The function $G(z) := \tau(z) - h(z\tau(z))$ is analytic on $D_{<1}$. After Proposition 3.2, $\tau = \tau(z)$ fulfills the fixed point equation $\tau = h(z\tau)$ on $D_{\rho_-}$. Hence $G(z) = 0$ on $D_{\rho_-}$ and, consequently, $G(z)$ is identically zero on the whole set $D_{<1}$. Since $h(0) \neq 0$, $\tau(z)$ cannot have a zero in $D_{<1}$. ▌

**Remark 3.6.** The fixed point equation above is equivalent to the third degree equation (written in terms of $w(z)$),

$$w(1 + \zeta w)^2 - z(\zeta + w)^2 = 0 \tag{3.18}$$

Solving this equation and identifying the adequate roots would furnish the explicit expression of $w$ as a function of $z \in D_{<1}$. Unfortunately this explicit form is too complicated to be useful.

The bound $|w(z)| < 1$ holds on $D_{<1}$. Therefore,

$$I^{<}(z) := \lim_{N \to \infty} I_N(z) = \lim_{N \to \infty} p_1 \cdots p_N r(w_N(z)) = ar(w(z)) \tag{3.19}$$

is analytic on $D_{<1}$. Since $I_N(z) = -I_N(z^{-1})$ for any $N \in \mathbb{N}$ we conclude that, for $z \in D_{>1}$, we have $I^{>}(z) := \lim_{N \to \infty} I_N(z) = -I^{<}(z^{-1})$ which is analytic on $D_{>1}$. This completes the proof of Theorem 1.1. ▌

According to Theorem 1.1, the singularities of $m(z)$, if they exist, must be localized in the unit circle $S^1$. In particular, $F_N(z)$ may have singularities at points $z$ for which $w_n(z) + 1 = 0$ for some $0 \leqslant n \leqslant N$. As we will next see, these singularities exist indeed and are poles. We will call these singularities the Lee–Yang singularities of the magnetization at the origin $m(z)$.

## 4. THE LEE–YANG SINGULARITIES OF *m*

For $n \in \mathbb{N}$ we are interested in solutions of the following equation

$$w_n(z) = -1 \tag{4.1}$$

Since $w_n(z)$, as a function of $z$, maps both sets $D_{<1}$ and $D_{>1}$ into itself, the solutions of (4.1), if they exist, must lie on $S^1 := \{z \in \mathbb{C};\ |z| = 1\}$. To show the existence of solutions we observe that (4.1) is equivalent to the polynomial equation $zP_n(z) + Q_n(z) = 0$, which has at least 1 and at most $2^{n+1} - 1$ distinct solutions in $\mathbb{C}$. By the previous argument these solutions must all lie in $S^1 \subset \mathbb{C}$. We will establish below that the number of distinct solutions is precisely $2^{n+1} - 1$.

Let us denote by $\mathfrak{S}_n \subset S^1$ the set of all solutions of (4.1) for a given $n \in \mathbb{N}$. Since $\overline{w_n(z)} = w_n(\bar{z})$ we conclude that if $z_0 \in \mathfrak{S}_n$ then $\bar{z}_0 \in \mathfrak{S}_n$ as well. Since $w_n(-1) = -1$, it is also clear that $-1 \in \mathfrak{S}_n$.

Writing $z = e^{i\phi}$, $0 \leqslant \phi \leqslant 2\pi$ for the elements of $S^1$, let us enumerate the elements of $\mathfrak{S}_n$ in increasing order according to the angle $\phi$. Define

$$\mathfrak{R}_n := \{\phi(n, l) \in [0, \pi],\ 1 \leqslant l \leqslant 2^n\}$$

where $z(n, l) = e^{i\phi(n, l)}$ satisfies $w_n(z(n, l)) = -1$. Since $z = -1$ is always a solution of this equation we have $\phi(n, 2^n) = \pi$. In words, the set $\mathfrak{R}_n$ is, for each given $n$, the set of phases of all Lee–Yang singularities contained in the upper half unit circle counted in increasing order, i.e., $\phi(n, l) \geqslant \phi(n, l')$ for $l > l'$.

In the next two subsections we will establish the following theorem:

**Theorem 4.1.** The set $\mathfrak{R}_n$ becomes a dense set on $[0, \pi]$ when $n \to \infty$ for $r \leqslant \zeta \leqslant 1/3$. ▮

To start with the analysis, let us look in more detail at the sequence $w_n(z)$, $n \in \mathbb{N}$, for $z \in S^1$. Firstly, consider the function $h(w)$ for $w \in S^1$. Since $1/w = \bar{w}$ we can write

$$h(w) = w^2 \left(\frac{1 + \zeta\bar{w}}{1 + \zeta w}\right)^2$$

For $w =: e^{i\theta}$ we have $h(w) =: e^{iL(\theta)}$, with

$$L(\theta) = L(\zeta, \theta) := 2\theta - 4 \arctan\left(\frac{\zeta \sin\theta}{1 + \zeta\cos\theta}\right) \tag{4.2}$$

Hence, defining $\phi$ through $z =: e^{i\phi}$ and $\phi_j$ through $w_j(z) =: e^{i\phi_j}$, $j \in \mathbb{N}$, and taking into account that $w_{j+1}(z) = zh(w_j(z))$, we have

$$\begin{aligned} \phi_0 &:= \phi, & \phi &\in [0, \pi] \\ \phi_{i+1} &:= \phi_0 + L(\phi_i), & i &\in \mathbb{N} \end{aligned} \tag{4.3}$$

These equations will be treated as a discrete dynamical system on $\mathbb{R}$.

### 4.1. Properties of the Discrete Dynamical System

**Proposition 4.2.** For all $x \in \mathbb{R}$ and $\zeta \in [0, 1)$, $L$ is a continuous monotone increasing function of $x$ such that $L'(x) \geqslant L'(0) \equiv \lambda(\zeta)$, where

$$\lambda(\zeta) := 2\left(\frac{1-\zeta}{1+\zeta}\right) \tag{4.4}$$

Moreover, the function $L$ has the following properties:

(i) $L(x + 2\pi) = 4\pi + L(x)$, $x \in \mathbb{R}$,

(ii) $L''(x) \geqslant 0$, for $x \in [0, \pi]$,

(iii) $\lim_{\zeta \uparrow 1} L(x) = \begin{cases} 0, & 0 \leqslant x < \pi \\ 2\pi, & x = \pi \\ 4\pi, & \pi < x \leqslant 2\pi \end{cases}$ ▮

*Proof.* By definition

$$L'(x) = 2\left(\frac{1-\zeta^2}{1 + 2\zeta \cos x + \zeta^2}\right) \geqslant \lambda(\zeta) \tag{4.5}$$

This relation says that $L$ is a strictly positive, strictly increasing function in $\mathbb{R}_+ \backslash \{0\}$. We also have

$$L''(x) = \frac{4\zeta(1-\zeta^2) \sin x}{(1 + 2\zeta \cos x + \zeta^2)^2} \geqslant 0 \tag{4.6}$$

for $0 \leqslant x < \pi$. Note that, for $\zeta = 1$ and $0 \leqslant x < \pi$, $L$ satisfies the differential equation $L'(c) = 0$ with initial data $L(0) = 0$. This shows (iii) since $L(\pi) = 2\pi$ for any $\zeta$. Item (i) is clear. ▮

**Corollary 4.3.** For $x \in \mathbb{R}_+$ and $\zeta \in [0, 1]$ one has the simple bound $L(x) \geqslant \lambda(\zeta)\, x$. ▮

In view of Proposition 4.2 and since $\lambda(\zeta) > 1$ for $0 \leqslant \zeta < 1/3$, the Liapunov exponent of the discrete dynamical system (4.3) is positive in this range of $\zeta$. We have also the following

**Lemma 4.4.** If $\phi_0 \in \mathbb{R}_+$ then $\phi_i \in \mathbb{R}_+$ for all $i \in \mathbb{N}$, $i \geqslant 0$. ▮

*Proof.* Let us assume, by an induction argument, that $\phi_i \in \mathbb{R}_+$. By Corollary 4.3 one has

$$\phi_{i+1} \geqslant \phi_0 + \lambda(\zeta)\, \phi_i \geqslant 0 \tag{4.7}$$

by the hypothesis. Since this holds for all $i \in \mathbb{N}$ the lemma is proven. ▮

From now on we will consider only starting points in $\mathbb{R}_+$, what reduces the dynamical system to a dynamical system on $\mathbb{R}_+$. Let us denote by $\phi_i(\phi)$ the $i$th iterated of the discrete dynamical system having $\phi$ as starting point.

**Lemma 4.5.** Let $\phi^2$ and $\phi^1$ two starting points with $\phi^2 > \phi^1 \geqslant 0$. Denote $\phi_i^a = \phi_i(\phi^a)$, with $i \in \mathbb{N}$ and $a = 1, 2$. Then for all $i \in \mathbb{N}$ one has $\phi_i^2 > \phi_i^1 \geqslant 0$. ▮

*Proof.* Assume, by an induction argument, that $\phi_i^2 > \phi_i^1$. Then

$$\begin{aligned} \phi_{i+1}^2 - \phi_{i+1}^1 &= (\phi^2 - \phi^1) + \int_{\phi^1}^{\phi^2} L'(s)\, ds \\ &\geqslant (\phi^2 - \phi^1) + \lambda(\zeta)(\phi_i^2 - \phi_i^1) > 0 \end{aligned} \tag{4.8}$$

by the hypothesis. ▮

Note that $\phi_i(\phi)$ is a continuous function of the starting point $\phi$, since it is a finite composition of continuous functions. This fact and the previous lemma immediately imply:

**Lemma 4.6.** For all $i \in \mathbb{N}$, $\phi_i(\phi)$ is a strictly increasing continuous function on $\mathbb{R}_+$. ▮

This lemma has an important consequence:

**Corollary 4.7.** The sets $\mathfrak{R}_n$, $n \in \mathbb{N}$, are composed by $2^n$ distinct elements, what in particular says that (4.1) has, for each $n \in \mathbb{N}$, precisely $2^{n+1} - 1$ distinct solutions in $S^1$. Each set $\mathfrak{R}_n$ can be ordered such that $0 \leqslant \phi(n, l) < \phi(n, l') \leqslant \pi$ for $l < l'$ and the elements $\phi(n, l)$ are such that

$$\phi_n(\phi(n, l)) = \pi + 2\pi(l - 1) \tag{4.9}$$

for $1 \leqslant l \leqslant 2^n$. All these facts hold for all $\zeta \in [0, 1)$. ▮

*Proof.* The sets $\mathfrak{R}_n$, $n \in \mathbb{N}$, defined above can also be characterized as the sets of all $\phi(n, l) \in [0, \pi]$, $1 \leqslant l \leqslant 2^n$, for which $\phi_n(\phi(n, l)) = \pi + 2\pi n_l$ for some $n_l \in \mathbb{N}$, depending on $l$. To see that $n_l = l - 1$ we note that, due to Lemma 4.6, $\phi_n(\phi)$ is a strictly increasing continuous function of $\phi$ with $\phi_n(0) = 0$ and $\phi_n(\pi) = (2^{n+1} - 1)\pi$. Therefore, $\phi_n\colon [0, \pi] \to [0, (2^{n+1} - 1)\pi]$ is an invertible map, the inverse being also strictly increasing and continuous. We can define $\phi(n, l) := \phi_n^{-1}(\pi + 2\pi(l - 1))$ for $1 \leqslant l \leqslant 2^n$. Clearly $\phi(n, l) < \phi(n, l')$ if $l < l'$.

### 4.2. The Sets $\mathfrak{R}_n$ in the Ferromagnetic region $(0 \leqslant \zeta \leqslant 1/3)$

In this subsection we will show that, in the ferromagnetic region $(0 \leqslant \zeta \leqslant 1/3)$, the Lee–Yang singularities become dense in the whole unit circle $S^1$ when the thermodynamic limit is taken.

The next two theorems are crucial for understanding the behavior of the Lee–Yang singularities when $n$ goes to infinity.

**Theorem 4.8.** With the above definitions we have

$$\lim_{n \to \infty} (\phi(n, l+1) - \phi(n, l)) = 0 \tag{4.10}$$

uniformly in $l$, for all $0 \leqslant \zeta \leqslant 1/3$. ▌

*Proof.* Call $\phi_i(n, l) := \phi_i(\phi(n, l))$, the $i$th iteration started from the point $\phi(n, l)$. We have for $0 \leqslant i \leqslant n-1$

$$\begin{aligned} &\phi_{i+1}(n, l+1) - \phi_{i+1}(n, l) \\ &\quad = \phi(n, l+1) - \phi(n, l) + L(\phi_i(n, l+1)) - L(\phi_i(n, l)) \\ &\quad \geqslant \phi(n, l+1) - \phi(n, l) + \lambda(\zeta)(\phi_i(n, l+1) - \phi_i(n, l)) \\ &\quad \geqslant \left( \sum_{a=0}^{i} \lambda(\zeta)^a \right) (\phi(n, l+1) - \phi(n, l)) \end{aligned} \tag{4.11}$$

by Lemma 4.5, by (4.5) and by induction. Since $\phi_n(n, l+1) - \phi_n(n, l) = 2\pi$, we get

$$\phi(n, l+1) - \phi(n, l) \leqslant 2\pi \left( \sum_{a=0}^{n-1} \lambda(\zeta)^a \right)^{-1} \tag{4.12}$$

uniformly in $l$. For $\lambda(\zeta) \geqslant 1$, what happens if $0 \leqslant \zeta \leqslant 1/3$, the limit (4.10) follows. ▌

We can also prove the following theorem:

**Theorem 4.9.** With the above definitions we have

$$\lim_{n \to \infty} \phi(n, 1) = 0 \tag{4.13}$$

for all $0 \leqslant \zeta \leqslant 1/3$. ▌

*Proof.* For $0 \leqslant i \leqslant n-1$ we have

$$\phi_{i+1} = \phi(n, 1) + L(\phi_i) \geqslant \phi(n, 1) + \lambda(\zeta)\, \phi_i \tag{4.14}$$

$$\geqslant \left( \sum_{a=0}^{i} \lambda(\zeta)^a \right) \phi(n, 1) \tag{4.15}$$

Hence, taking $i = n-1$,

$$\phi(n, 1) \leqslant \pi \left( \sum_{a=0}^{n-1} \lambda(\zeta)^a \right)^{-1} \tag{4.16}$$

from which the theorem follows. ▌

Theorems 4.8 and 4.9, together with the fact that $\pi \in \mathfrak{R}_n$ for all $n \in \mathbb{N}$, imply Theorem 4.1. ▌

**Remark 4.10.** Notice that both Theorems 4.8 and 4.9 hold also at the critical point $\zeta = 1/3$.

### 4.3. The Sets $\mathfrak{R}n$ in the Paramagnetic Region $(1/3 < \zeta \leqslant 1)$

In Section 5 we will see that, for each value of $\zeta$ in the paramagnetic region $(1/3 < \zeta \leqslant 1)$, the sets $\mathfrak{S}_n$ are excluded from a neighborhood of $z = 1$. However, one should expect that, in analogy to the situation in the ferromagnetic region, the sets $\mathfrak{S}_n$ become dense in some proper subset of $S^1$ when the limit $n \to \infty$ is taken.

In this subsection we will prove a weaker statement, namely that there exists a proper subset $A_\kappa$ of $S^1$ such that $\mathfrak{S}_n \cap A_\kappa$, becomes dense on $A_\kappa$ when $n \to \infty$. Let us start capturing the main ingredients for the proof, as we learned from the last subsection.

The proofs presented in the previous subsection show clearly that the phenomenon of the Lee–Yang singularities becoming dense is closely related to the positivity of the Liapunov exponents of our discrete dynamical system. In order to obtain analogous results for the region $1/3 < \zeta \leqslant 1$ one has to take into account that, for these values of $\zeta$, the derivative $L'(x)$ is not larger than 1 for all points $x \in [0, 2\pi]$. However, if the trajectory of the dynamical system visits frequently enough regions where $L'$ is large, a positive lower bound for the Liapunov exponent can be obtained with few iterations. This can be the case if, for instance, the points of the trajectory successively jump from a region where $L' < 1$ to other where $L' > 1$, large enough. Below we will follow this sort of idea in order to advance into the paramagnetic region $1/3 < \zeta \leqslant 1$.

Consider the sets $\mathfrak{R}_n$ for $1/3 < \zeta \leqslant 1$. For our discrete dynamical system we have

$$\phi_{i+2} = \phi_0 + L(\phi_0 + L(\phi_i)), \qquad i \geqslant 1 \tag{4.17}$$

Define, for $\phi \in [0, \pi]$ and $x \in \mathbb{R}_+$, $M(\phi, x) := L(\phi + L(x))$. Since

$$M_x(\phi, x) = L'(\phi + L(x))\, L'(x) \tag{4.18}$$

one has $M_x(\phi, x + 2\pi) = M_x(\phi, x)$, $x \in \mathbb{R}$.

Assume that, for some convenient set $[\kappa^0, \pi] \subset [0, \pi]$, $0 \leqslant \kappa^0 < \pi$, one has

$$M^0 := \inf_{\phi \in [\kappa^0, \pi]} \inf_{x \in \mathbb{R}} M_x(\phi, x) \geqslant 1 \tag{4.19}$$

This basic assumption will be proven at the end of this subsection. Let us first draw some of its general consequences. For $\kappa^0 \leqslant a < b \leqslant \pi$ and $0 \leqslant x < y$ on has

$$\begin{aligned} M(b, y) - M(a, x) &= (M(b, y) - M(b, x)) + (M(b, x) - M(a, x)) \\ &= \int_x^y M_x(b, s)\, ds + \int_a^b M_\phi(\mu, x)\, d\mu \end{aligned} \tag{4.20}$$

But $M_\phi(\mu, x) = L'(\mu + L(x)) \geqslant 0$ and so

$$M(b, y) - M(a, x) \geqslant \int_x^y M_x(b, s)\, ds \geqslant M^0(y - x) \tag{4.21}$$

with $M^0$ given by (4.19). Under the assumption (4.19) and using (4.21) we have, as in (4.11),

$$\begin{aligned} &\phi_{i+2}(n, l+1) - \phi_{i+2}(n, l) \\ &\quad = (\phi(n, l+1) - \phi(n, l)) + M(\phi(n, l+1), \phi_i(n, l+1)) - M(\phi(n, l), \phi_i(n, l)) \\ &\quad \geqslant \phi(n, l+1) - \phi(n, l) + M^0(\phi_i(n, l+1) - \phi_i(n, l)) \\ &\quad \geqslant \left[ \sum_{a=0}^{i/2} (M^0)^a \right] (\phi(n, l+1) - \phi(n, l)) \end{aligned} \tag{4.22}$$

for $1 \leqslant i \leqslant n-2$, $i$ even, provided $\phi(n, l)$, $\phi(n, l+1) \in \mathfrak{R}_n \cap [\kappa^0, \pi]$. This gives

$$\phi(n, l+1) - \phi(n, l) \leqslant 2\pi \left( \sum_{a=0}^{n/2} (M^0)^a \right)^{-1} \tag{4.23}$$

for $n$ even and, following analogous steps,

$$\phi(n, l+1) - \phi(n, l) \leqslant \left( \frac{3-\zeta}{1+\zeta} \right) 2\pi \left( \sum_{a=0}^{(n-1)/2} (M^0)^a \right)^{-1} \tag{4.24}$$

for $n$ odd.

Since, by the hypotheses, the right hand side goes to zero when $n \to \infty$ uniformly in $l$, we have proven that, under the circumstances that provide (4.19), the set $\mathfrak{R}_n \cap [\kappa^0, \pi]$ becomes dense on itself when $n \to \infty$. To establish that $\mathfrak{R}_n \cap [\kappa^0, \pi]$ becomes indeed dense over the whole interval $[\kappa^0, \pi]$ we will also need the following result.

**Theorem 4.11.** Let $l^0 = l^0(n)$ be given by $\phi(n, l^0) = \min_\phi \{\phi \in \mathfrak{R}_n : \phi > \kappa^0\}$. Then, the limit $\lim_{n \to \infty} \phi(n, l^0) = \kappa^0$. ∎

*Proof.* Let us consider the dynamical system (4.17) with $\phi_0$ replaced by $\kappa^0$: $\kappa_{i+2} = \kappa^0 + M(\kappa^0, \kappa_i)$. As in (4.22), we write

$$\begin{aligned} \phi_{i+2}(n, l^0) - \kappa_{i+2} &= \phi(n, l^0) - \kappa^0 + M(\phi(n, l^0), \phi_i(n, l^0)) - M(\kappa^0, \kappa_i) \\ &\geqslant \phi(n, l^0) - \kappa^0 + M^0(\phi_i(n, l^0) - \kappa_i) \end{aligned} \tag{4.25}$$

The inequality in (4.25) follows from (4.21).

Repeating the steps (4.22)–(4.24) we conclude the proof of Theorem 4.11. Recall only that $\phi(n, l^0) > \kappa^0 \geqslant \phi(n, l^0 - 1)$ implies $\phi_j(n, l^0) > \kappa_j \geqslant \phi_j(n, l^0 - 1)$, giving $\phi_n(n, l^0) - \kappa_n \leqslant 2\pi$. ∎

We shall next establish that $\mathfrak{R}_n \cap [\kappa^0, \pi]$ becomes indeed dense over the whole interval $[\kappa^0, \pi]$ under the above assumptions when $n \to \infty$. After the above results, we only need to show that the sets $\mathfrak{R}_n \cap [\kappa^0, \pi]$ contain other elements than $\pi$ for the considered values of $\zeta$ and for n large enough.

Since we already know that, for $\zeta = 1/3$, $\mathfrak{R}_n$ becomes dense on $S^1$ when $n$ goes to infinite, we can argue that, for $\zeta \equiv \zeta_1$ larger but close enough to 1/3, there is a fixed $n_0$ large enough such that $\mathfrak{R}_{n_0} \cap [\kappa^0, \pi] \backslash \{\pi\} \neq \varnothing$. This is because the elements of $\mathfrak{R}_{n_0}$ depend continuously on $\zeta$. We have seen that $\mathfrak{R}_n \cap [\kappa^0, \pi]$ becomes dense on itself for $n \to \infty$ and by Theorem 4.11, we conclude that for $\zeta \equiv \zeta_1$ the set $\mathfrak{R}_n \cap [\kappa^0, \pi]$ becomes dense on $[\kappa^0, \pi]$ when $n \to \infty$. This sort of argument can, of course, be repeated for larger values of $\zeta$ covering the whole interval $1/3 < \zeta < \zeta_l$.

It remains now only to prove that the basic assumption (4.19) indeed holds for suitable values of $\zeta$. For this a careful analysis of our discrete dynamical system is necessary.

Let us introduce the sets $\mathscr{B} := \{x \in \mathbb{R}: \lambda(\zeta)\, L'(x) > 1\}$ and $\mathscr{R} := \mathbb{R} \backslash \mathscr{B}$, where $\lambda(\zeta)$ was defined in (4.4). By (4.2) these definitions say that $L'(x)\, L'(y) > 1$ for any $y \in \mathbb{R}$ provided $x \in \mathscr{B}$.

Let us first show that the set $\mathscr{B}$ is non-empty if $\zeta \in (1/3, 1]$. A simple computation shows that $\mathscr{B} = (\gamma, 2\pi - \gamma) + 2\pi\mathbb{Z}$, where $\gamma = \gamma(\zeta)$ is the solution in $[0, \pi]$ of the equation $\cos(\gamma) = g(\zeta)$, where

$$g(\zeta) := \begin{cases} 1, & \text{if } \ 0 \leqslant \zeta \leqslant 1/3 \\ \dfrac{3\zeta^2 - 8\zeta + 3}{2\zeta}, & \text{if } \ 1/3 < \zeta \leqslant 1 \end{cases} \tag{4.26}$$

Notice that $|g(\zeta)| \leqslant 1$, as one easily checks. This says that $\mathscr{B}$ is a non-empty set for $1/3 \leqslant \zeta \leqslant 1$. One checks easily too that $g(\zeta)$ is a continuous and strictly decreasing function for $1/3 \leqslant \zeta \leqslant 1$.

From (4.18) and (4.2) we conclude that, for $x \in \mathscr{B}$, one has $M_x(\phi, x) > 1$ for any $\phi \in [0, \pi]$.

What happens if $x \in \mathscr{R}$? We will next show that for $x \in \mathscr{R}$ one has $\phi + L(x) \in \mathscr{B}$, provided $1/3 < \zeta < \zeta_l$, where $\zeta_l \simeq 0.46409$, and $\phi$ is large enough. Therefore, under these circumstances we also have the bound $M_x(\phi, x) > 1$. We express these results in the following theorem:

**Theorem 4.12.** There exist a constant $\zeta_l$, $1/3 < \zeta_l < 1$, $\zeta_l \simeq 0.46409$, and a continuous strictly increasing function $K(\zeta)$ on the interval $[1/3, \zeta_l]$ with $K(1/3) = 0$, $K(\zeta_l) = \pi$, such that, for $\zeta \in [1/3, \zeta_l)$ and $\phi \in (K(\zeta), \pi]$, one has $M_x(\phi, x) > 1$ for any $x \in [0, 2\pi]$. ▮

This theorem says that we can adopt $\kappa^0 = K(\zeta)$ in our previous analysis.

*Proof of Theorem 4.12.* As we have already discussed, the case $x \in \mathscr{B}$ is trivial, by the definitions. For $x \in \mathscr{R}$ we want to show that, under the hypotheses, one has $\phi + L(x) \in \mathscr{B}$, i.e.,

$$\gamma < (\phi + L(x)) \bmod 2\pi < 2\pi - \gamma \tag{4.27}$$

We will split the proof in two parts. The first for $x \in [0, \gamma]$ and the second for $x \in [2\pi - \gamma, 2\pi]$.

Denote

$$
\begin{aligned}
K(\zeta) &:= \gamma(\zeta) + L(\zeta, \gamma(\zeta)) \\
&= 3 \arccos\left(\frac{3 - 8\zeta + 3\zeta^2}{2\zeta}\right) - 4 \arctan\left(\frac{\sqrt{3(3-\zeta)(3\zeta-1)}}{5 - 3\zeta}\right)
\end{aligned} \tag{4.28}
$$

where $L$ was defined in (4.2). After some computations, it can be shown directly from (4.28) that for $\zeta \in (1/3, 1]$,

$$
K'(\zeta) = \frac{1}{\zeta} \sqrt{\frac{3(3-\zeta)}{3\zeta - 1}} > 0
$$

Therefore, $K$ is a continuous strictly increasing function of $\zeta$ in the interval $[1/3, 1]$ with $K(1/3) = 0$ and $K(1) = 5\pi/3 > \pi$. This proves the existence of a unique $\zeta_l \in (1/3, 1)$ with $K(\zeta_l) = \pi$. A numerical computation indicates that $\zeta_l \simeq 0.46409$.

1. The case $x \in [0, \gamma]$.

Here we want to show that

$$
\gamma < \phi + L(x) < 2\pi - \gamma \tag{4.29}
$$

Since $L \geqslant 0$, the first inequality is clear since, by the hypotheses, $\phi > K(\zeta) = \gamma(\zeta) + L(\zeta, \gamma(\zeta)) \geqslant \gamma(\zeta) - L(x)$.

For the second inequality in (4.29) we notice that for $x \in [0, \gamma]$ one has $L(x) \leqslant L(\zeta, \gamma(\zeta))$ since $L$ is an increasing function. Hence, the inequality $\gamma + \phi + L(x) < 2\pi$ holds if $\gamma(\zeta) + L(\zeta, \gamma(\zeta)) < \pi$, since $\phi \leqslant \pi$. But this inequality is always true for $1/3 \leqslant \zeta < \zeta_l$. This completes the proof of the Theorem 4.12 for $x \in [0, \gamma]$.

2. The case $x \in [2\pi - \gamma, 2\pi]$.

For $x \in [2\pi - \gamma, 2\pi]$ we write $x = 2\pi - y$, where $y \in [0, \gamma]$ and notice that $L(x) = 4\pi - L(y)$. We want to show that

$$
\gamma < \phi - L(y) < 2\pi - \gamma \tag{4.30}
$$

The second inequality is trivial, because $\phi + \gamma - L(y) < 2\pi$, since $\phi$ and $\gamma$ lie in $[0, \pi]$ and $L \geqslant 0$. The inequality $\gamma + L(y) < \phi$ will hold if $\phi > \gamma(\zeta) + L(\zeta, \gamma(\zeta)) = K(\zeta)$, which is one of the conditions of Theorem 4.12 and can always be satisfied provided $1/3 \leqslant \zeta < \zeta_l$. This completes the proof of the Theorem 4.12. ∎

## 5. ANALYTICITY OF $m$ AROUND $z=1$ FOR $1/3<\zeta<1$

We shall now determine the domain of analyticity of the magnetization $m$ in the paramagnetic phase. We aim to show that, for $1/3<\zeta<1$ and $\mathbf{p}\in\pi$, $m$ can be analytically continued through the arc $S^1\backslash A_\kappa$, which, in view of Theorem 1.1, proves the analytic statement of Theorem 1.2.

We begin by showing some preliminary results.

**Theorem 5.1.** Given $1/3<\zeta\leqslant 1$, let $\bar{\phi}=\bar{\phi}(\zeta)$ and $\kappa=\kappa(\zeta)$ be given by (1.11). Then, for all $\phi_0<\kappa$, the fixed point equation

$$\phi=\phi_0+L(\phi) \tag{5.1}$$

admits a unique stable solution $\phi^*=\phi^*(\zeta,\phi_0)$ to which the dynamical system (4.3) converges. Moreover, the solution $\phi^*$ is a monotonically increasing function of the initial condition $\phi_0$ with $0<\phi^*<\bar{\phi}\leqslant\pi$. ∎

**Remark 5.2.** Theorem 5.1 implies that the arc $S^1\backslash A_\kappa$ is free of Lee–Yang singularities since (4.1) can never be satisfied for any $n\in\mathbb{N}$.

*Proof.* By Proposition 4.2, $L$ is convex on the domain $0\leqslant x<\pi$ with $L'(0)<1$ if $1/3<\zeta\leqslant 1$. Therefore, there always exists $\kappa=\kappa(\zeta)>0$ such that, for all $\phi_0\leqslant\kappa$ the graph of $\phi_0+L(x)$ intercepts the graph of $x$ at least once in this domain. The constant $\kappa$ is obtained by imposing $\bar{\phi}=\kappa+L(\bar{\phi})$ and the tangency condition

$$L'(\bar{\phi})=2\left(\frac{1-\zeta^2}{1+2\zeta\cos\bar{\phi}+\zeta^2}\right)=1 \tag{5.2}$$

These equations determine uniquely $\kappa$ and the fixed point solution $\bar{\phi}$ to (5.1) at $\phi_0=\kappa$, in accordance to Theorem 5.1.

Next we will show that, the iterates of the discrete map (4.3) converges to the fixed point $\phi^*$. Let $\phi^*=\phi^*(\zeta,\phi_0)$ be the smallest solution to (5.1). It follows from Proposition 4.2 that $\phi<\phi_0+L(\phi)<\phi^*<\bar{\phi}$ holds for all $\phi\leqslant\phi^*$ and $\phi_0<\kappa$. In particular, using (5.1), $\phi_0=\phi^*-L(\phi^*)<\phi^*$. So, (4.3) maps the interval $[0,\phi^*]$ into itself and it is contractive in this domain:

$$|L(\phi)-L(\sigma)|\leqslant L'(\phi^*)\ |\phi-\sigma|\equiv q^*\ |\phi-\sigma|$$

for all $\phi,\sigma[0,\phi^*]$. This implies convergence of the sequence $\{\phi_j\}_{j\geqslant 0}$ to the solution $\phi^*$ since $q^*<L'(\bar{\phi})\equiv 1$ and concludes the proof of Theorem 5.1. ∎

We are now in position to determine the domain of analyticity of the magnetization $m$ according to Theorem 1.2. We begin by showing the following result.

**Lemma 5.3.** For $1/3<\zeta<1$ and $\mathbf{p}\in\pi$, the sequence $\{m_N = F_N + I_N,\ N\in\mathbb{N}\}$ converges to a continuous function $\tilde{m}=\tilde{F}+\tilde{I}$ on $S^1\backslash A_\kappa$, where $\kappa=\kappa(\zeta)$ is given by (1.11). ∎

*Proof.* Let $z=e^{i\phi_0}$ with $\phi_0\in[0,\pi)$ and recall that $e^{i\phi_j}=w_j(z)$, $j\in\mathbb{N}$, are such that $\phi_j$ satisfy the discrete map (4.3). By Theorem 5.1, we have

$$|r(w_j)|=\left(\frac{1-\cos\phi_j}{1+\cos\phi_j}\right)^{1/2}\leqslant\sqrt{\frac{2}{\delta}} \tag{5.3}$$

for some $\delta=\delta(\zeta,\phi_0)>0$ provided $1/3<\zeta\leqslant 1$ and $0\leqslant\phi_0<\kappa$. As a consequence, for any $\varepsilon>0$,

$$|F_M(z)-F_N(z)|\leqslant\sum_{j=N}^{M-1}a_j\,|r(w_j)|\leqslant\sqrt{\frac{2}{\delta}}\sum_{j=N}^{M-1}a_j\leqslant\varepsilon$$

for $M>N$, $N$ large enough, since $\{a_n\}_{n\in\mathbb{N}}$ is summable. This says that the sequence of continuous functions $F_N$ converges uniformly to a continuous function $\tilde{F}$ on $S^1\backslash A_\kappa$ with $\kappa=\kappa(\zeta)$ for all $1/3<\zeta\leqslant 1$.

Analogously, under the same conditions, we have

$$|I_M(z)-I_N(z)|\leqslant 2\left|\frac{w_N-w_M}{(1+w_M)(1+w_N)}\right|\leqslant\frac{1}{\delta}\,|w_N-w_M|\leqslant\varepsilon$$

for any $\varepsilon>0$ provided $M$ and $N$ are large enough, as a consequence of Theorem 5.1. Thus, the sequence of continuous functions $I_N$, $N\in\mathbb{N}$, converges uniformly to a continuous function $\tilde{I}$ on $S^1\backslash A_\kappa$. This concludes the proof of Lemma 5.3. ∎

By using the "Edge-of-the-Wedge" theorem, the analytic function $m$ studied in Section 1.1 can be analytically continued through the arc $S^1\backslash A_\kappa$ provided $\tilde{m}$ is the limit of $m$ when $z$ approaches the arc $S^1\backslash A_\kappa$. We shall establish this in the following theorem.

**Theorem 5.4.** Let $1/3<\zeta\leqslant 1$ and $\mathbf{p}\in\pi$. Let $m=m(\zeta,z,\mathbf{p})$ be the analytic functions on $z$ stated in Theorem 1.1 for the domain $\{|z|<1\}\cup\{|z|>1\}$. If $z_0\in S^1\backslash A_\kappa$, we have

$$\lim_{z\to z_0}|m(z)-\tilde{m}(z_0)|=0 \quad ∎$$

*Proof.* Let $w_j = w_j(z)$ be given by (2.3) with $z \in \mathbb{C}$, and recall the sequence $w_j$, $j \in \mathbb{N}$, satisfies

$$w_j = zh(w_{j-1}) \equiv z(h \circ w_{j-1}) \tag{5.4}$$

with $w_0 = z$. We set $w_{0,j} := w_j(z_0)$, $j \in \mathbb{N}$, with $z_0 \in S^1 \backslash A_\kappa$ and note that one can write $w_{0,j} = e^{i\phi_j}$ with the sequence $\phi_j$, $j \in \mathbb{N}$, satisfying (4.3).

We have $|w_0 - w_{0,0}| = |z - z_0|$ and for $j = 1, 2, \ldots$,

$$\begin{aligned} |w_{j+1} - w_{0,j+1}| &= |zh(w_j) - z_0 h(w_{0,j})| \\ &\leqslant |z - z_0| + |z| \, |h(w_j) - h(w_{0,j})| \\ &= |z - z_0| + |zQ(w_j, w_{0,j})| \, |w_j - w_{0,j}| \end{aligned} \tag{5.5}$$

where $Q(w, w') \equiv Q(1, w, w')$ is defined in (3.14).

For fixed $\varepsilon > 0$ and $0 < q < 1$ we let $z \in \mathbb{C}$ be such that

$$|z - z_0| < \varepsilon(1 - q) \tag{5.6}$$

Note that, by Theorem 5.1 and Proposition 4.2,

$$|Q(w_{0,j}, w_{0,j})| = \frac{2(1-\zeta^2)}{1 + 2\zeta \cos \phi_j + \zeta^2} = L'(\phi_j) < 1$$

holds for all $j \in \mathbb{N}$ since $\phi_j < \bar{\phi}$ (recall (5.2)). Now, using the fact that $Q(w, w')$ is a continuous function of $w$ and $w'$, there always exist numbers $0 < q < 1$ and $\varepsilon > 0$ such that, for $z$ satisfying (5.6), we have

$$\sup_{w \in S^1 \backslash A_\kappa} \sum_{w' : |w' - w| \leqslant \varepsilon} |zQ(w', w)| < q \tag{5.7}$$

We now claim that $\lim_{z \to z_0} |w_j - w_{0,j}| = 0$ holds for any $j \in \mathbb{N}$. Clearly, $|w_0 - w_{0,0}| < \varepsilon$. By induction, let us assume that $|w_j - w_{0,j}| < \varepsilon$ for some $j \in \mathbb{N}$. Then $|zQ(w_j, w_{0,j})| < q$ and, from (5.5) and (5.6), we get

$$|w_{j+1} - w_{0,j+1}| \leqslant |z - z_0| + q \, |w_j - w_{0,j}| < \varepsilon$$

uniformly in $j \in \mathbb{N}$. Since $\varepsilon$ is arbitrary, this establishes the claim.

If $|z - z_0| < (1 - q) \, \varepsilon\delta/2$ with $\delta$ as in (5.3), we have

$$D^j(z, z_0) := |r(w_j) - r(w_{0,j})| = \frac{2 \, |w_j - w_{0,j}|}{|(1 + w_j)(1 + w_{0,j})|} < \varepsilon$$

and consequently

$$|m_N(z) - \tilde{m}_N(z_0)| \leqslant |F_N - \tilde{F}_N| + |I_N - \tilde{I}_N|$$
$$\leqslant \sum_{j=0}^{N-1} a_j D^j(z, z_0) + p_1 \cdots p_N D^N(z, z_0) < \varepsilon$$

holds uniformly in $N$. This concludes the proof of Theorem 5.4 and the proof of the analytic statement of Theorem 1.1. ▌

## 6. CONTINUITY AND DIFFERENTIABILITY OF $m$

The aim of this section is to establish Theorem 1.6. The quenched magnetization at origin $m$ will be here considered as a real valued function of $z: m: \mathbb{R}_+ \mapsto \mathbb{R}$.

Our study on the smoothness of the quenched magnetization $m$ will be divided in three subsections. In the first it will be examined under which conditions $m$ is discontinuous at $z = 1$ and its value will be computed. The second subsection will establish smoothness of $F$ under the assumption of Theorem 6.5 which will be proven in the last subsection.

### 6.1. Jump of the Magnetization

This subsection is devoted to show that, for $z \in \mathbb{R}_+$ and $0 \leqslant \zeta < 1/3$, $m = \lim_{n \to \infty} m_N$ is a continuous function of $z$ if $\mathbf{p} \in \pi_0$ and discontinuous at $z = 1$ if $\mathbf{p} \in \pi_a$ with $a > 0$.

We start by noting the following facts about the function $r$ (the proof will be omitted):

**Proposition 6.1.** $r: x \in \mathbb{R}_+ \mapsto r(x) \in \mathbb{R}$ given by (3.7) is a continuous monotone decreasing function of $x$ with $r(x) = -r(1/x) \leqslant 1$. ▌

Proposition 6.1 and (3.11) imply $m(z) = -m(1/z)$ and since $\sum_{n=0}^{N-1} a_n + p_1 \cdots p_N = 1$, we have $-1 \leqslant m(x) \leqslant 1$. In addition, $m(1) = 0$ as a consequence of $w_n(1) = 1$ for all $n \in \mathbb{N}$.

Let $\zeta \in [0, 1/3)$, $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi_0$. By Proposition 6.1 and recalling that $\lim_{n \to \infty} p_1 \cdots p_n = 0$, we have

$$|m_N - m_M| \leqslant \sum_{n=M+1}^{N-1} a_n |r(w_n)| + p_1 \cdots p_M |1 - p_{M+1} \cdots p_N| \, |r(w_N)| < \varepsilon \tag{6.1}$$

for any $\varepsilon > 0$ provided $N$ and $M$, $M < N$, are large enough. The sequence $\{m_N\}_{N \geqslant 0}$ is thus a uniform Cauchy sequence of continuous functions which establishes the continuity of $m$ in $z \in \mathbb{R}_+$.

Note that this statement is also true for the sequence $\{F_N\}_{N \geqslant 0}$ if $\pi_0$ is replaced by $\pi$ in the assumptions. So, in order $m$ to be discontinuous at $z = 1$, $I_N$ must converge, as $N \to \infty$, to different limits, $I^>$ and $I^<$, depending on whether $z$ is larger or smaller than 1. This requires $\mathbf{p} \in \pi_a$ with $a > 0$ and $\zeta \in [0, 1/3)$.

By definition $w_n(z), n \in \mathbb{N}$, satisfy (5.4) with $w_0 = z$. The following proposition gives some properties of $h$ which will be useful for this and the next subsection. Its proof follows by an easy computation and will be omitted.

**Proposition 6.2.** For $\zeta \in [0, 1/3)$, $h\colon w \in \mathbb{R}_+ \mapsto h(w) \in \mathbb{R}_+$ given by (2.1) is a continuous monotone increasing function of $w$, bounded from below and above by $\zeta^2$ and $1/\zeta^2$, respectively, with $h(1) = 1$. Its first derivative

$$h'(w) = \frac{2(\zeta + w)(1 - \zeta^2)}{(1 + \zeta w)^3}$$

has a maximum value at $\hat{w} = (1 - 3\zeta^2)/2\zeta > 1$ given by

$$h'(\hat{w}) = \frac{8}{27\zeta(1 - \zeta^2)}$$

with $h''(w) > 0$ for $0 \leqslant w < \hat{w}$ and $h''(w) < 0$ for $w > \hat{w}$. In addition, $h'(1) = 2(1 - \zeta)/(1 + \zeta)$ is strictly larger than 1 for all $0 \leqslant \zeta < 1/3$. ∎

We shall now describe in the next proposition the limit points of $w_n$. By (3.11), we can restrict ourselves to $z \in [0, 1)$. This implies, in view of Corollary 2.2, that $w_n \in [0, 1)$ for all $n \in \mathbb{N}$.

**Proposition 6.3.** Let $\zeta \in [0, 1/3)$ and $z \in [0, 1)$. The sequence $w_n, n \in \mathbb{N}$, converges to the unique solution $w = w(\zeta, z)$ of the fixed point equation $w = zh(w)$ in $[0, 1)$. In addition, $w$ is a monotone increasing function of $z$ with

$$\underline{w} := \lim_{z \nearrow 1} w = \frac{1 - 2\zeta - \zeta^2}{2\zeta^2} - \frac{1 - \zeta^2}{2\zeta^2}\left(\frac{1 - 3\zeta}{1 + \zeta}\right)^{1/2} \quad ∎ \tag{6.2}$$

*Proof.* The convergence statement of Proposition 6.3 follows from Theorem 3.4 and Corollary 2.2. The fixed point $w = w(\zeta, z)$ is monotone in

$z$ because $h$ is a monotone increasing function. The fixed point equation $w = h(w)$ at $z = 1$ is equivalent to

$$(w-1)[\zeta^2 w^2 - (1 - 2\zeta - \zeta^2)\, w + \zeta^2] = 0$$

which, when $0 \leqslant \zeta < 1/3$ has three real solutions but only one in $[0, 1)$.

**Remark 6.4.** Proposition 6.3 can be extended to any $z \in \mathbb{R}_+$ by using the relation $w(\zeta, z) = 1/w(\zeta, 1/z)$ so, $\bar{w} := \lim_{z \searrow 1} w(\zeta, z) = 1/\underline{w}$.

We now turn to the magnetization jump. As a corollary of the result established in (6.1) we have $\lim_{z \nearrow 1} \lim_{N \to \infty} F_N = 0$ for all $\mathbf{p} \in \pi$. Therefore, in view of Proposition 6.3, for $\mathbf{p} \in \pi_a$ with $a > 0$, we have

$$\lim_{z \nearrow 1} \lim_{N \to \infty} m_N = \lim_{z \nearrow 1} \lim_{N \to \infty} I_N = ar(\underline{w})$$

Analogously, by Proposition 6.1 and (3.11),

$$\lim_{z \searrow 1} \lim_{N \to \infty} m_N = ar(\bar{w}) = -ar(\underline{w})$$

from where (1.13) follows, since (recall (6.2))

$$r(\underline{w}) = \frac{(1-\zeta)^{1/2}(1+\zeta)^{1/2}}{1-\zeta}$$

This proves the continuity and the magnetization jump statements of Theorem 1.6. ▌

## 6.2. DIFFERENTIABILITY OF $F$

We have shown that $F$ is a continuous function of $z$ for all $z \in \mathbb{R}_+$ and $\mathbf{p} \in \pi$. Here it will be established the differentiability statement of Theorem 1.6 under the same conditions. We recall that, in view of Theorem 1.2, we need only to consider $\zeta \in [0, 1/3)$.

We shall use very often the recursion relation (5.4) with the initial condition $w_0 = z$. Hence, to show smoothness of $F$, we need to estimate the $k$th derivative, $D^k(f \circ g)$, of the composition of two functions $f$ and $g$. If $\mathscr{P}_k$ denotes the set of all partitions $\mathbf{P} = (P_1, \dots, P_s)$ of $\{1, 2, \dots, k\}$ and $|P|$ denotes the cardinality of the set $P$, we have

$$D^k(f \circ g) = \sum_{\mathbf{P} \in \mathscr{P}_k} \left( \prod_{j=1}^{s} D^{|P_j|} g \right) f^{(s)} \circ g \tag{6.3}$$

Now, let us assume that $f$ and $g$ are smooth functions such that, for all $n \in \mathbb{N}$,

$$\frac{1}{n!}\,|f^{(n)}(x)| \leqslant C_f \kappa_f^n \qquad \text{and} \qquad \frac{1}{n!}\,|g^{(n)}(x)| \leqslant C_g \kappa_g^n \tag{6.4}$$

hold for some finite constants $C_f$, $\kappa_f$, $C_g$ and $\kappa_g$.

Then, since $\sum_j |P_j| = k$ for any partition $\mathbf{P} \in \mathscr{P}_k$, (6.3) can be bounded by

$$\begin{aligned}\frac{1}{k!}\,|D^k(f \circ g)| &\leqslant \frac{1}{k!} \sum_{\mathbf{P} \in \mathscr{P}_k} \left( \prod_{j=1}^{s} |P_j|!\; C_g \kappa_g^{|P_j|} \right) s!\; C_f \kappa_f^s \\ &= C_f \kappa_g^k R_k(C_g \kappa_f) \end{aligned}\tag{6.5}$$

where

$$R_k(x) := \frac{1}{k!} \sum_{\mathbf{P} \in \mathscr{P}_k} \left( \prod_{j=1}^{s} |P_j|! \right) s!\; x^s \tag{6.6}$$

To continue the evaluation of (6.5) we need an estimate on $R_k$. It will be convenient to write (6.6) in a more suitable form. If $n_j$ denotes the number of times a set $P$ with $|P| = j$ occurs in the partition $\mathbf{P}$, $R_k$ can be written as

$$R_k(x) = \sum_{\mathbf{n}} \frac{(\sum_j n_j)!}{n_1! \cdots n_k!}\, x^{\sum_j n_j} \tag{6.7}$$

where the summation is over all $k$-component vectors $\mathbf{n} = (n_1, ..., n_k) \in \mathbb{N}^k$ such that $n_1 + 2n_2 + \cdots + kn_k = k$.

The following theorem is our main technical result of this section.

**Theorem 6.5.** Given $k \in \mathbb{N}_+$, let $R_k\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the polynomial defined by (6.7). Then,

$$R_k(x) \leqslant x(1+x)^{k-1} \quad \blacksquare \tag{6.8}$$

Theorem 6.5 will be proven in the next subsection.

Applying Theorem 6.5 to the inequality (6.5), gives an upper bound of the form (6.4):

$$\frac{1}{k!}\,|D^k(f \circ g)| \leqslant C_{f \circ g} \kappa_{f \circ g}^k \tag{6.9}$$

where

$$C_{f\circ g}=C_f\frac{C_g\kappa_f}{1+C_g\kappa_f} \qquad \text{and} \qquad \kappa_{f\circ g}=\kappa_g(1+C_g\kappa_f) \tag{6.10}$$

The estimate (6.9) can be used to control inductively derivatives of $w_n$ in (5.4) provided $f$ is replaced by $h$ and $g$ replaced by $w_{n-1}$. The next lemma controls derivative of arbitrary order of $h$.

**Lemma 6.6.** Let $\zeta\in[0, 1/3)$ and $w\in\mathbb{R}_+$ such that

$$w\leqslant\frac{1-2\zeta^2}{\zeta^3}\left[1+\frac{1-\zeta^2}{(1-2\zeta^2)^{1/2}}\right] \tag{6.11}$$

Then, the following holds

$$\frac{1}{k!}|h^{(k)}(w)|\leqslant(h'(w))^k \quad \blacksquare \tag{6.12}$$

for all $k\in\mathbb{N}_+$.

*Proof.* For $k\in\mathbb{N}_+$, we write $h=a\cdot b$ where $a=:(\zeta+w)^2$ and $b=:(1+\zeta w)^{-2}$ and use the Leibniz formula:

$$h^{(k)}=\sum_{p=0}^{k}\binom{k}{p}a^{(k)}b^{(k-p)}$$

where derivatives of $b$ can be explicitly evaluated:

$$b^{(j)}(w)=(-1)^j(j+1)!\frac{\zeta^j}{(1+\zeta w)^{2+j}} \qquad j=0, 1, \cdots$$

We thus have

$$h^{(k)}=(-1)^k k!\left(\frac{\zeta}{1+\zeta w}\right)^k c_k \tag{6.13}$$

where

$$c_k=\frac{1-\zeta^2}{\zeta^2(1+\zeta w)^2}[(k-1)(1-\zeta^2)-2\zeta(\zeta+w)]$$

By induction, equation (6.12) is implied if

$$\frac{1}{k!}\,|h^{(k)}(w)| \leqslant \frac{1}{(k-1)!}\,|h^{(k-1)}(w)| \cdot h'(w)$$

holds for $k \geqslant 2$. In view of (6.13), this follows provided

$$\begin{aligned}&\zeta(1+\zeta w)^2\,|(k-1)(1-\zeta^2)-2\zeta(\zeta+w)|\\ &\quad\leqslant 2(1-\zeta^2)(\zeta+w)|(k-2)(1-\zeta^2)-2\zeta(\zeta+w)|\end{aligned} \tag{6.14}$$

There are three cases to be considered:

**Case I** $(k-1)(1-\zeta^2)>(k-2)(1-\zeta^2)>2\zeta(\zeta+w)$,
**Case II** $(k-1)(1-\zeta^2)>2\zeta(\zeta+w)\geqslant(k-2)(1-\zeta^2)$ and
**Case III** $2\zeta(\zeta+w)\geqslant(k-1)(1-\zeta^2)>(k-2)(1-\zeta^2)$.

We can verify that (6.14) is true in all cases provided the following condition holds:

$$\zeta(1+\zeta w)^2 \leqslant 2(1-\zeta^2)(\zeta+w) \tag{6.15}$$

For case I and II, simply replace $(k-1)$ by $(k-2)$ in the left-hand side of (6.14). For case III, note that (6.14) is equivalent to

$$\begin{aligned}&[\zeta(1+\zeta w)^2-2(1-\zeta^2)(\zeta+w)][2\zeta(\zeta+w)-(k-1)(1-\zeta^2)]\\ &\quad\leqslant 2(1-\zeta^2)(\zeta+w)\end{aligned}$$

which is always verified since, under the conditions of case III and (6.15), the left-hand side cannot be positive.

Solving (6.15) for $w$ gives the upper bound (6.11). This concludes the proof of Lemma 6.6. ∎

We are now ready to prove the following

**Theorem 6.7.** Given $0 \leqslant \zeta < 1/3$, let $z \in \mathbb{R}_+$ be such that $w = w(\zeta, z)$, the solution to the equation $w = zh(w)$, satisfies (6.11). Then, for $n, k \in \mathbb{N}$ with $k \geqslant 1$, there exist finite constants $C = C(\zeta, z)$ and $\mu = \mu(\zeta, z)$, such that

$$\frac{1}{k!}\,|D^k w_n| \leqslant \begin{cases} C^n \mu^k & \text{if } z \neq 1 \\ C\mu^{nk} & \text{if } z = 1 \end{cases} \tag{6.16}$$

holds with $C(\zeta, z) < 1$ if $z \neq 1$ and $\mu(\zeta, 1) = \lambda(\zeta)$ given by (4.4). ∎

*Proof.* Theorem 6.7 will be proven by induction. Let us denote

$$W_n^k := \frac{1}{k!} |D^k w_n| \qquad k \in \mathbb{N} \tag{6.17}$$

($W_n^0 \equiv w_n$) where $D$ denotes the differential operator $d/dz$, and suppose

$$W_n^k \leqslant W_n \mu_n^k \tag{6.18}$$

holds with positive constants $W_n$ and $\mu_n$, $\mu_n \geqslant 1$. Since $w_0 = z$, we have

$$W_0^j \leqslant \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6.19}$$

which certainly satisfies (6.18) with $W_0 = 2$ and $\mu_0 = 1$ (these are chosen so for late convenience). Differentiating (5.4) $k$-times and taking absolute value, gives

$$\frac{1}{k!} |D^k w_{n+1}| \leqslant z \frac{1}{k!} |D^k(h \circ w_n)| + \frac{1}{(k-1)!} |D^{k-1}(h \circ w_n)|$$

Applying Eqs. (6.9) and (6.10) with $f$ replaced by $h$ and $g$ replaced by $w_n$, and using Lemma 6.6, yields

$$W_{n+1}^k \leqslant \left( z + \frac{1}{\mu_n} \right) \frac{\eta_n w_n}{1 + \eta_n w_n} [\mu_n (1 + \eta_n w_n)]^k$$

where $\eta_n := h'(w_n)$. Here, the positivity of $W_n$ and $\mu_n \geqslant 1$ have been taking into account. This gives, in view of (6.18), the following recursive relations

$$X_{n+1} = \eta_n \left( z + \frac{1}{\mu_n} \right) \frac{X_n}{1 + X_n} \qquad \text{and} \qquad \mu_{n+1} = \mu_n (1 + X_n) \tag{6.20}$$

where $X_n := \eta_n W_n$.

We now distinguish two possible scenarios: either $X_n$ tends to zero as $n \to \infty$ or remains bounded from above and below by finite positive constants. We shall see that $X_n \to 0$ exponentially fast if $0 \leqslant \zeta < 1/3$ and $z \neq 1$ and this leads $\mu_n \to \mu < \infty$ whereas, if $0 \leqslant \zeta < 1/3$ and $z = 1$, $X_n$ converges to a finite positive constant and $\mu_n \to \infty$ exponentially fast. To show this we need the following result.

**Lemma 6.8.** Let $\zeta \in [0, 1/3)$, $z \in \mathbb{R}_+$ as in the Theorem 6.7 and let $\eta_n = \eta_n(\zeta, z) := h'(w_n)$. Then, the sequence $\eta_n$, $n \in \mathbb{N}$, converges to $\eta$ as $n \to \infty$ with $0 \leqslant \eta z < 1$ if $z \neq 1$. If $z = 1$, $\eta_n = \lambda(\zeta) = 2(1-\zeta)/(1+\zeta) > 1$ for all $n \in \mathbb{N}$. ∎

*Proof.* From Proposition 6.3 and the continuity of $h'$, $\eta_n$ converges to a limit $\eta = h'(w)$ where $w = w(\zeta, z)$ is the solution of the fixed point equation $w = zh(w)$, provided $\zeta \in [0, 1/3)$ and $z \in [0, 1)$. From Remark 6.4 this can also be extended to $z \in (1, \infty)$. Differentiating $w = zh(w)$ with respect to $z$, we obtain

$$h'(w)\, z = 1 - \left(\frac{\partial w}{\partial z}\right)^{-1} h(w) \tag{6.21}$$

where $\partial w/\partial x$, in view of Proposition 6.3, is strictly positive establishing that $\eta z < 1$. Note that $w_n = 1$ for all $n \in \mathbb{N}$ provided $z = 1$ (recall Corollary 2.2) and this implies $\eta_n = \lambda(\zeta)$ concluding the proof of the lemma. ∎

We shall prove that $X_n \to 0$ and $\mu_n \to \mu < \infty$ by contradiction. We let $\zeta \in [0, 1/3)$ and $z \in \mathbb{R}_+$, $z \neq 1$, and assume that $\mu_n$ diverges as $n \to \infty$. Then, in view of Lemma 6.8 and our hypothesis, there exists a finite number $n_0 = n_0(\zeta, z)$, such that

$$\eta_n \left(z + \frac{1}{\mu_n}\right) < 1$$

for all $n > n_0$. This leads, by (6.20), the sequence $\{X_n\}_{n \in \mathbb{N}}$ to converge to zero exponentially fast and the sequence $\{\mu_n\}_{n \in \mathbb{N}}$ to converge to $\mu < \infty$, contradicting our assumption.

We now let $\zeta \in [0, 1/3)$ and $z = 1$. Then, by Lemma 6.8

$$\alpha_n := \eta_n \left(z + \frac{1}{\mu_n}\right) = \lambda \left(1 + \frac{1}{\mu_n}\right) > 1$$

and this implies, by induction, that $1 + X_n > \alpha_n > 1$ for all $n \in \mathbb{N}$, and consequently

$$\mu_{n+1} = \prod_{j=0}^{n} (1 + X_j) > \prod_{j=0}^{n} \alpha_n$$

leads $\mu_n \to \infty$. For the induction, note that $1 + X_0 = 1 + \eta_0 W_0 = 1 + 2\lambda > \alpha_0$. Assuming that $1 + X_n > \alpha_n$ is true, we have, in view of (6.20),

$$1 + X_{n+1} = 1 + \alpha_n - \frac{\alpha_n}{1 + X_n} > \alpha_n > \alpha_{n+1}$$

since the sequence $\mu_n$, $n \in \mathbb{N}$, is monotone increasing.

Since the $\mu_n$ tends to infinity exponentially fast so do $\alpha_n$ tend to $\lambda$. As a consequence, there exist a constant $C_1$ such that

$$\mu_{n+1} = \prod_{j=0}^{n} (1 + X_j) \leqslant C_1 \lambda^{n+1}$$

This concludes the proof of Theorem 6.7. ∎

We shall now finish the proof of Theorem 1.6.

*Proof of Theorem 1.6* (Conclusion). Let us fix $z = 1$. Differentiating (3.8) $k$-times, gives

$$\frac{1}{k!} D^k F_N = \sum_{n=1}^{N-1} a_n \frac{1}{k!} D^k[r \circ w_n] \tag{6.22}$$

which can be estimated by using (6.9), Theorem 6.7 and

$$\frac{1}{j!} |r^{(j)}(w_n)| \leqslant \frac{2}{2^j} \tag{6.23}$$

Here, (6.23) is obtained from the explicit formula $r^{(j)}(x) = (-1)^j j!\, 2(1+x)^j$, $j \in \mathbb{N}$, and from Corollary 2.2.

Now, (6.9) with $f$ replaced by $r$ and $g$ replaced by $w_n$, implies

$$\frac{1}{k!} |D^k[r \circ w_n]| \leqslant 4C \left(1 + \frac{C}{2}\right)^{k-1} \mu^{nk} \tag{6.24}$$

with $C$ and $\mu$ as in Theorem 6.7.

In view of (6.22) and (6.24), there exist finite constants, $C_1$ and $C_2$, such that

$$\frac{1}{k!} |D^k F_N - D^k F_M| \leqslant C_1 C_2^k \sum_{n=M+1}^{N} a_n \mu^{kn} < \varepsilon$$

for any $\varepsilon > 0$, provided $\{a_n \mu^{kn}\}_{n \geqslant 0}$ is a summable sequence and $M, N > N_0$ for some $N_0 = N_0(\zeta, \mathbf{p}, k) < \infty$ large enough. The sequence $\{D^K F_N\}_{N \geqslant 0}$, for any $k \in \mathbb{N}$, is a uniform Cauchy sequence of continuous function of $z$ in $\mathbb{R}_+$. This concludes the proof of Theorem 1.6. ∎

### 6.3. Polynomial Domination of $R_k$

*Proof of Theorem 1.6.* Let us first assume that the following upper bound

$$R_k(x) \leqslant x \sum_{j=0}^{k-1} R_j(x) \tag{6.25}$$

holds for all $x \in \mathbb{R}_+$ and $k \geqslant 1$, with $R_0(x) \equiv 1$. The inequality (6.25) will be established afterward.

We prove (6.8) by induction. For $k=1$, $R_1(x) \leqslant x$ by (6.25). Now, assume (6.8) valid for $k=1,..., n$, $n \in \mathbb{N}_+$. From (6.25), we have

$$R_{n+1}(x) \leqslant x\left(1 + x \sum_{j=1}^{n} (1+x)^{j-1}\right) = x(1+x)^n \quad \blacksquare \tag{6.26}$$

*Proof of (6.25).* Define

$$y_\ell = \sum_{j=1}^{\ell} n_j, \qquad \ell = 1,..., k$$

and notice that the coefficients of $R_k$ can be written as

$$\frac{(\sum_j n_j)!}{n_1! \dots n_k!} = \frac{y_1!}{n_1!} \frac{y_2!}{n_2! y_1!} \cdots \frac{y_k!}{n_k! y_{k-1}!} = \binom{y_2}{n_2} \cdots \binom{y_k}{n_k}$$

With this notation, (6.7) can thus be written as

$$R_k(x) = \sum_{n_1} x^{n_1} \sum_{n_2} \binom{y_2}{n_2} x^{n_2} \cdots \sum_{n_k} \binom{y_k}{n_k} x^{n_k} \, \delta_k(n_1 + \cdots + k n_k) \tag{6.27}$$

where $\delta_k(m) = 1$ if $m = k$ and 0 otherwise.

For $1 \leqslant i < j \leqslant k$, let $R_k^{[i,j]}(x)$ be the polynomial of order $k$ in $x$ obtained from (6.27) by setting $n_1,..., n_{i-1}, n_{j+1},..., n_k$ all equal to 0:

$$R_k^{[i,j]}(x) = \sum_{n_i} x^{n_i} \sum_{n_{i+1}} \binom{y_{i+1}}{n_{i+1}} x^{n_{i+1}} \cdots \sum_{n_j} \binom{y_j}{n_j} x^{n_j} \, \delta_k(i n_i + \cdots + j n_j)$$

Now, note that $n_k = 0, 1$ are the only values which satisfy the equation $n_1 + 2n_2 + \cdots + k n_k = k$. In the case $n_k = 1$, we have $n_1 = \cdots = n_{k-1} = 0$ and $y_k = 1$. Equation (6.27) can thus be written as

$$R_k(x) = R_k^{[1,k-1]}(x) + x = R_k^{[2,k-2]}(x) + x R_{k-1}(x) + x \tag{6.28}$$

Here, $R_k^{[1,k-1]}(x)$ has been written as

$$R_k^{[1,k-1]}(x) = R_k^{[2,k-2]}(x) + \sum_{n_1 \neq 0} x^{n_1} \cdots \sum_{n_{k-1}} \binom{y_{k-1}}{n_{k-1}} \times x^{n_{k-1}} \delta_k(n_1 + \cdots + (k-1)\, n_{k-1})$$

and, by making the change of variables $n_1' = n_1 - 1$, the second term can be written as $xR_{k-1}$. For the first term, we note that $2n_2 + \cdots + (k-1)\, n_{k-1} = k$ holds only with $n_{k-1} = 0$. This implies (6.28).

This procedure can be repeated for the sum over $n_2$ in $R_k^{[2,k-2]}(x)$. For the term with $n_2 \geqslant 1$, we change variable to $n_2' = n_2 - 1$

$$x \sum_{n_2 \geqslant 1} x^{n_2 - 1} \cdots \sum_{n_{k-1}} \binom{y_{k-2}}{n_{k-2}} x^{n_{k-2}} \times \delta_{k-2}(2n_2 - 2 + \cdots + (k-2)\, n_{k-2}) = xR_{k-2}^{[2,k-2]}$$

and notice that $R_{k-2}^{[2,k-2]} \leqslant R_{k-2}$ by the positivity of $x$.

Relation (6.28) therefore says

$$R_k(x) \leqslant R_k^{[3,k-3]}(x) + x(R_{k-2} + R_{k-1}(x) + 1)$$

where we have used that $3n_3 + \cdots + (k-2)\, n_{k-2} = k$ holds only with $n_{k-2} = 0$.

Continuing this process, we get

$$R_k(x) \leqslant x(R_{[k/2]}(x) + \cdots + R_{k-1}(x) + 1)$$

where $[v]$ is the integer part of $v \in \mathbb{R}$. Since $R_j(x) \geqslant 0$ for $x \in \mathbb{R}_+$, this concludes the proof of (6.25).

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